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CAN A WORMHOLE SUPPORTED BY ONLY SMALL AMOUNTS OF EXOTIC MATTER REALLY BE TRAVERSABLE?

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Abstract. Recent studies have shown that: (a) quantum effects may be sufficient to support a wormhole throat, and (b) the total amount of "exotic matter" can be made arbitrarily small. Unfortunately, using only small amounts of exotic matter may result in a wormhole that flares out too slowly to be traversable in a reasonable length of time. Combined with the Ford-Roman constraints, the wormhole may also come close to having an event horizon at the throat. This Brief Report examines a model that overcomes these difficulties, while satisfying the usual traversability conditions. This model also confirms that the total amount of exotic matter can indeed be made arbitrarily small.

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1. Introduction

Wormholes may be defined as handles or tunnels linking different universes or widely separated regions of our own universe. That such wormholes may be traversable by humanoid travelers was first conjectured by Morris and Thorne [1]. To hold such a wormhole open, violations of certain energy conditions are unavoidable [1, 2, 3, 4]. As a result, the energy density of matter may be seen as negative by some observers. Morris and Thorne called such matter "exotic."

While all classical forms of matter obey the weak energy condition (WEC) $T_{\alpha\beta}\mu^\alpha\mu^\beta \geq 0$ for all time-like vectors and, by continuity, all null vectors, quantum fields can generate locally negative energy densities, which may be arbitrarily large at a given point. In addition to the WEC, wormhole spacetimes violate the averaged null energy condition (ANEC) [2, 5], which states that $\int T_{\alpha\beta}k^\alpha k^\beta d\lambda \geq 0$, where the integral is taken along a complete null geodesic with tangent vector k and affine parameter λ. While quantum field theory has generously allowed the existence of exotic matter, it also constrains the wormhole geometries, as a detailed analysis by Ford and Roman [5] has shown. In particular,

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the exotic matter has to be confined to a shell very much thinner than the throat.

Consider now the spherically symmetric line element

$$ds^2 = -e^{2\gamma(r)}dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{1}$$

Here $\gamma(r)$ is the *redshift function* and $b(r)$ the *shape function*. Proposals to restrict the exotic matter to an arbitrarily thin region are discussed by Kuhfittig [6, 7] under the condition that $b'(r)$ be close to unity near the throat. A more general discussion of an arbitrarily small energy condition violation is presented by Visser *et al.* [8] in conjunction with the main results of global analysis in classical general relativity. Assuming the ANEC violation, the integral representing the total amount of exotic matter is shown to be arbitrarily small, provided that $e^{\gamma(r_0)} \to 0$, where $r = r_0$ is the throat of the wormhole. The limit actually refers to a sequence of wormholes. (If $e^{\gamma(r)} \to 0$ as $r \to r_0$ for a specific wormhole, we would be dealing with an event horizon.)

Finally, Hochberg *et al.* [9] in discussing their self-consistent wormhole solution of semiclassical gravity, present numerical evidence suggesting that quantum effects may be sufficient to support a wormhole throat.

The purpose of this Brief Report is to show that a wormhole supported by only minute amounts of exotic matter may be traversable in practice, not just in principle. We assume that the Ford-Roman constraints are satisfied, while avoiding an event horizon at the throat. The wormhole is small enough to be traversed in a reasonable length of time; the radial tidal forces and the gradient of the redshift function are small enough to accommodate a humanoid traveler. That only small amounts of exotic matter are needed is confirmed in Sec. 3.

2. The ANEC violation

In place of the line element (1) we will use the form

$$ds^2 = -e^{2\gamma(r)}dt^2 + e^{2\alpha(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{2}$$

To save space we will omit the usual discussion of the basic wormhole features except to note that the graph of the function $\alpha = \alpha(r)$ has a vertical asymptote at $r = r_0$: $\lim_{r\to r_0+} \alpha(r) = +\infty$. (For further details see Refs. [1, 7].) From the line elements above we have

$$b(r) = r\left(1 - e^{-2\alpha(r)}\right). \tag{3}$$

One of the general bounds for wormhole geometries discussed in Ref. [5], Sec. V, is shown to be weakest when $b'(r)$ is close to unity near the

throat. Accordingly, we assume that the graph of $\alpha = \alpha(r)$ is steep enough near $r = r_0$ to meet this condition. If $b'(r)$ is close to unity near the throat, then the embedding diagram will flare out very slowly. This slow flaring out need not be fatal, however, as shown in Ref. [7]. (We will return to this point in the next section.)

Also from Ref. [7] the WEC violation in terms of α and γ is given by $\rho - \tau < 0$, where

$$\rho - \tau = \frac{1}{8\pi}\left[\frac{2}{r}e^{-2\alpha(r)}\left(\alpha'(r) + \gamma'(r)\right)\right]. \tag{4}$$

Following Visser *et al.* [8], a natural way to measure the mass of the wormhole (including both asymptotic regions) is

$$\int_{\text{Vol}} \rho(r)dV = 2\int_{r_0}^{\infty} 4\pi r^2 \rho(r)dr. \tag{5}$$

Because of the ANEC violation, our interest centers mainly on the integral $\int_{\text{Vol}}(\rho - \tau)dV$. Since Eq. (4) can be written

$$\rho - \tau = \frac{1}{8\pi r^2}\left[2re^{-2\alpha(r)}(\alpha'(r) + \gamma'(r))\right],$$

it follows from Eq. (5) that

$$\int_{\text{Vol}} (\rho - \tau)dV = 2\int_{r_0}^{\infty} re^{-2\alpha(r)}(\alpha'(r) + \gamma'(r))dr. \tag{6}$$

Integrating by parts,

$$\int_{\text{Vol}} (\rho - \tau)dV = -2\int_{r_0}^{\infty} (\alpha(r) + \gamma(r))e^{-2\alpha(r)}(1 - 2r\alpha'(r))dr \tag{7}$$

since the boundary term vanishes at the throat due to the factor $e^{-2\alpha(r)}$ and at infinity due to the asymptotic behavior.

A good choice for $\alpha(r)$ is

$$\alpha(r) = \frac{K^n}{(r - r_0)^n}, \quad n \geq 1,$$

for some constant K having the same units as r. The condition $n \geq 1$ ensures that $b'(r) \approx 1$ near the throat. To avoid an event horizon, we let

$$\gamma(r) = -\frac{L^n}{(r - r_2)^n}, \quad n \geq 1,$$

for some constant L and where r_2 is such that $0 < r_2 < r_0$.

To satisfy the Ford-Roman constraints, the WEC must be satisfied outside the interval $[r_0, r_1]$ for some r_1. To accomplish this, construct

α and γ so that $|\alpha'(r_1)| = |\gamma'(r_1)|$. Now let $\gamma_1(r) = -\gamma(r)$, choose a suitable K, and determine L so that $\alpha'(r) = \gamma_1'(r)$. The result is

$$L^n = \left[\frac{(r_1 - r_2)^{n+1}}{(r_1 - r_0)^{n+1}}\right] K^n. \tag{8}$$

With this choice of L it is easy to show that $|\alpha'(r)| > |\gamma'(r)|$ for $r_0 < r < r_1$; more precisely,

$$\alpha'(r) = -\frac{nK^n}{(r - r_0)^{n+1}} < -\frac{n(r_1 - r_2)^{n+1}K^n}{(r_1 - r_0)^{n+1}} \frac{1}{(r - r_2)^{n+1}} = \gamma'(r).$$

To the right of $r = r_1$ the inequality is reversed. As a result, we have $\rho - \tau < 0$ in the interval (r_0, r_1) and $\rho - \tau \geq 0$ for $r \geq r_1$. Also, since the exotic matter is confined to the spherical shell extending from $r = r_0$ to $r = r_1$, we have

$$\int_{\text{Shell}} (\rho - \tau) dV < 0, \tag{9}$$

which represents the "total amount" of energy-condition violating matter. One of our goals is to show that the integral (9) can be made arbitrarily small. To that end observe that by the mean-value theorem there exists a number $c \in (r_0, r_1)$ such that

$$\begin{aligned} \int_{\text{Shell}} (\rho - \tau) dV &= -2 \int_{r_0}^{r_1} (\alpha(r) + \gamma(r))\, e^{-2\alpha(r)} \left(1 - 2r\alpha'(r)\right) dr \\ &= -\frac{2}{r_1 - r_0} (\alpha(c) + \gamma(c)) e^{-2\alpha(c)} (1 - 2c\alpha'(c)) \\ &= -\frac{2}{r_1 - r_0} \left(\frac{K^n}{(c - r_0)^n} - \frac{L^n}{(c - r_2)^n}\right) e^{-2K^n/(c - r_0)^n} \\ &\qquad \times \left(1 + 2cnK^n \frac{1}{(c - r_0)^{n+1}}\right). \end{aligned} \tag{10}$$

To see why this integral may be vanishingly small, let $r_1 \to r_0$ and hence $c \to r_0$. By l'Hospital's rule the right side of Eq. (10) approaches 0. Unfortunately, since the construction of $\gamma(r)$ depends on r_1, this limit cannot be taken directly. In fact, for a fixed $\alpha(r)$, if $r_1 \to r_0$, $\gamma'(r_1)$ gets ever larger, causing the sequence $\gamma(r_0)$ would recede to $-\infty$, so that $e^{\gamma(r_0)} \to 0$, creating the very event horizon that we are trying to avoid.

We will return to this problem at the end of the next section.

3. Traversability conditions

As noted earlier, $b'(r)$ is close to unity near the throat. The resulting slow flaring out could make the wormhole too large to be traversable in a reasonable length of time. To analyze this problem, as well as the lateral tidal constraint and the gradient of the redshift function, we assume the following: with Eq. (8) in mind, suppose for now that r_2 has been chosen so that L is not much larger than K as long as n is small. That way we can use the same values for L and K and consider the case $L \neq K$ later. Also, since the wormholes are likely to be very large compared to r_0, we may assume that r_0, r_1, and r_2 are negligible for the purpose of estimating larger distances. As a result, $\gamma(r) = -\alpha(r)$; observe that the Ford-Roman constraints are trivially satisfied.

For our first model, we choose $n = 1$, so that $\alpha(r) = K/r$ and $\gamma(r) = -L/r = -K/r$. For the traversability conditions we follow Morris and Thorne [1]. The crucial radial tidal constraint is given by $|R_{\hat{1}'\hat{0}'\hat{1}'\hat{0}'}| = |R_{\hat{r}\hat{t}\hat{r}\hat{t}}| \leq (10^8\,\text{m})^{-2}$. By direct calculation or from Ref. [7]

$$(11) \quad |R_{\hat{r}\hat{t}\hat{r}\hat{t}}| = \left|e^{-2\alpha(r)}\left[\gamma''(r) - \alpha'(r)\gamma'(r) + (\gamma'(r))^2\right]\right| \\ = \left|e^{-2K/r}\left(-\frac{2K}{r^3} + \frac{2K^2}{r^4}\right)\right|.$$

The function on the right (inside the absolute value signs) attains respective minimum and maximum values at $r = \frac{1}{3}(3 \pm \sqrt{3})K$. To meet the constraint at these values, we choose $K = 5.0 \times 10^{-9}$ l.y. (light year).

Concerning the size of the wormhole as measured by the placement of the space stations, if we choose $r = 0.00006964$ l.y. $\approx 6.6 \times 10^8$ km, then

$$\gamma'(r) = \frac{K}{r^2} = \frac{5.0 \times 10^{-9} \times 9.46 \times 10^{15}\,\text{m}}{(0.00006964 \times 9.46 \times 10^{15}\,\text{m})^2} \approx 1.1 \times 10^{-16}\,\text{m}^{-1},$$

which meets the constraint $|\gamma'(r)| \leq g_{\oplus}/(c^2\sqrt{1 - b(r)/r})$. Finally, $b(r)/r$ is well within 1% of unity, also recommended in Ref. [1].

The distance $r = 6.6 \times 10^8$ km ≈ 4 A.U. may seem rather large, but if we assume, as suggested in Ref. [1], that the spaceship accelerates at $g_{\oplus} = 9.8\,\text{m/s}^2$ halfway to the throat and decelerates at the same rate until it comes to rest at the throat, then the throat would be reached in only about 6 days.

It is instructive to compare this model to one for which $n = 2$: let $\alpha(r) = K^2/r^2$ and $K = 1.0 \times 10^{-8}$ l.y. All the above conditions are

met, but the size of the wormhole is now only 0.000005789 l.y. $\approx \frac{1}{3}$ A.U.

For completeness let us momentarily suspend the condition that $b'(r)$ gets close to unity as we approach the throat: assume that $b(r) = r_0 = 2m$ outside a thin region extending from the throat to $r = a$, as in Ref. [8]; we further assume that $a < r_1$. Then

$$\alpha(r) = -\frac{1}{2}\ln(1 - \frac{2m}{r});$$

for the redshift function let $\gamma(r) = -K/r$. (Both K and m are measured in meters.) As in the previous section, we need to find $r = r_1$ such that

$$\alpha'(r_1) + \gamma'(r_1) = -\frac{m}{(1 - 2m/r_1)r_1^2} + \frac{K}{r_1^2} = 0,$$

whence

$$K = \frac{m}{1 - 2m/r_1}.$$

If r_{th} is the thickness of the shell, then $r_{\text{th}} = r_1 - r_0 = r_1 - 2m$, and

$$K = \frac{m}{1 - 2m/(2m + r_{\text{th}})} = m\left(2m\, r_{\text{th}}^{-1} + 1\right).$$

The factor r_{th}^{-1} will cause K to be large for any reasonable value of m. For example, Ref. [2], which discusses a wormhole based on the experimentally confirmed Casimir effect, gives $r_{\text{th}} = 10^{-12}$ m. As a consequence, the radial tidal constraint, Eq. (11), does not even come close to being satisfied at any point not too far from the throat. The primary reason is that the coefficient $e^{-2\alpha(r)}$ collapses to $1 - 2m/r$, which does not decay fast enough as $r \to 2m$. The resulting wormhole is therefore not traversable by humanoid travelers. (It is not hard to show that this conclusion holds for any differentiable function $\gamma = \gamma(r)$, particularly near $r = r_1$.)

In obtaining the estimates in this section, we made the simplifying assumption that $L = K$. An obvious alternative is to choose a smaller K to start with: referring to Eq. (8), if we let

$$A(n) = \frac{(r_1 - r_2)^{n+1}}{(r_1 - r_0)^{n+1}},$$

then $L^n = A(n)K^n$. (The expression for $A(n)$ shows that r_2 should be restricted so that $A(n)$ does not become excessively large.) Replacing K^n by $K^n_{\text{new}} = K^n/A(n)$ satisfies the requirements of Eq. (8) while keeping L^n and $\gamma(r)$ intact. As long as n is small, changing α to

$\alpha(r) = K_{\text{new}}^n/r^n$ will not have a drastic effect on the above estimates, as can be seen from Eq. (11).

For large n these estimates, as well as the resulting model, become clearly invalid: since $A(n)$ keeps increasing for any fixed r_2, $K_{\text{new}}^n = K^n/A(n)$ is a decreasing sequence. The exponential function $e^{-2\alpha(r)}$ will therefore begin to decay too slowly for the radial tidal constraint to be satisfied. (If the model were valid for large n, then the size of the wormhole could be decreased indefinitely, which would not make physical sense.)

Returning next to Eq. (10), if we model the wormhole by using the functions in this section, we can estimate the size of $\int_{\text{Shell}}(\rho - \tau)dV$ directly. Observe that, since $c < r_1$,

$$\left|\int_{\text{Shell}}(\rho - \tau)dV\right| = \left|-\frac{2}{r_1 - r_0}(\alpha(c) + \gamma(c))e^{-2\alpha(c)}(1 - 2c\alpha'(c))\right|$$
$$< \left|-\frac{2}{c - r_0}\left(\frac{K^n}{(c - r_0)^n} - \frac{L^n}{(c - r_2)^n}\right)e^{-2K^n/(c-r_0)^n} \times \left(1 + 2cnK^n\frac{1}{(c - r_0)^{n+1}}\right)\right|. \tag{12}$$

Simple calculator trials show that for any reasonable choice of r_1 (and hence of c),

$$\left|\int_{\text{Shell}}(\rho - \tau)dV\right| \ll 10^{-100},$$

even for $n = 1$. In fact, this extreme inequality holds even if L^n is much larger than K^n and r_1 many orders of magnitude larger than the value allowed by the Ford-Roman constraints.

While the conclusion seems to depend on our choices of α and γ, it is unlikely that any other acceptable choices would alter the results significantly, primarily because the basic features remain the same: both functions are assumed to be twice differentiable and hence continuous in their respective domains, $\gamma(r_0)$ is finite, and $\lim_{r \to r_0+}\alpha(r) = +\infty$. So by continuity, $\lim_{r_1 \to r_0}\gamma(r_1)e^{-2\alpha(r_1)} = 0$, even if $\gamma(r)$ assumes a completely different form from the one considered earlier, while

$$\lim_{r_1 \to r_0}\alpha(r_1)e^{-2\alpha(r_1)} = \lim_{r_1 \to r_0}\frac{\alpha(r_1)}{e^{2\alpha(r_1)}} = 0$$

by l'Hospital's rule, suggesting that the relevant quantities are indeed vanishingly small.

4. Conclusion

It is shown in this Brief Report that a wormhole held open with only small amounts of exotic matter may be traversable, not just in principle, but also in practice: in the model discussed the wormhole size permits traversal in a reasonable length of time, while satisfying the usual traversability conditions. The model also accommodates the Ford-Roman constraints—without introducing an event horizon at the throat. The integral measuring the total amount of exotic matter proved to be vanishingly small, confirming the results in Ref. [8].

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